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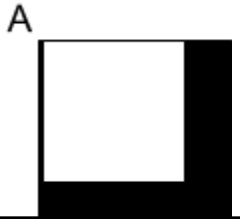
Exam for Grade 4

February 18, 2023

A SOLUTION	What number can replace the square to make the statement true? 5 x 11 = ☐ + 12
	Strategy: First evaluate the left side of the equation. $5 \times 11 = 55$, so $55 = \square + 12$. Then $\square = 55 - 12 = 43$. **To make the statement true, replace the square by 43.** ***Follow-Up**: Given $15 \times \square = \square + 84$. What one number can replace both squares to make the statement true?* [6]
B SOLUTION	In all, how many two-digit prime numbers have 4 as one of their digits?
	Strategy: Consider the ones and tens digits separately Any number with a ones digit of 4 is a multiple of 2 and is therefore not prime. If the tens digit is 4, the number could be 41, 43, 45, 47, or 49. But $45 = 9 \times 5$, and $49 = 7 \times 7$. The others are prime. **There are 3 two-digit prime numbers that have a digit of 4.** ***Follow-Up:** The sum of two prime numbers is 63. Find their product.* [122]
C SOLUTION	A rectangle has a perimeter of 2 meters and a length of 70 centimeters. Find the area of the rectangle in square centimeters.
	Strategy: Express all distances in the same unit of measure. The perimeter of the rectangle is 2 meters = 200 cm. Thus the sum of the length and width (called the semiperimeter) is 100 cm. Since the length is 70 cm, the width is 30 cm. **The area of the rectangle is** $70 \times 30 =$ **2100 sq cm.**
D SOLUTION	In the figure shown, two squares share corner A. The larger square has an area of 49 sq cm. The smaller square has an area of 25 sq cm. What is the perimeter of the shaded region, in cm? 

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	Strategy: Consider the ones and tens digits separately Any number with a ones digit of 4 is a multiple of 2 and is therefore not prime. If the tens digit is 4, the number could be 41, 43, 45, 47, or 49. But $45 = 9 \times 5$, and $49 = 7 \times 7$. The others are prime. **There are 3 two-digit prime numbers that have a digit of 4.** ***FOLLOW-UP:*** *The sum of two prime numbers is 63. Find their product.* [122]
E **SOLUTION**	In all, how many whole numbers between 400 and 600 are divisible by 9?
	METHOD 1: *Strategy: Find the number of multiples of 9 less than each given value.* Since $600 \div 9 = 66\frac{2}{3}$, there are 66 numbers less than 600 that are divisible by 9. Since $400 \div 9 = 44\frac{4}{9}$, there are 44 numbers less than 400 that are divisible by 9. Because neither 400 nor 600 is divisible by 9, **there are** $66 - 44 =$ **22 numbers between 400 and 600 that are divisible by 9.** **METHOD 2:** *Strategy: Split the numbers in the interval into groups of 9.* Between 400 and 600, there are 199 whole numbers, excluding 400 and 600. Dividing 199 by 9 yields 22 with a remainder of 1. Thus there are 22 complete groups of 9 consecutive numbers. Each group contains exactly one multiple of 9. The "leftover" number, 599, is not a multiple of 9, so there are 22 numbers in the interval that are divisible by 9. **METHOD 3:** *Strategy: Find the first number and repeatedly add 9.* $400 \div 9 = 44\frac{4}{9}$, so the first number greater than 400 that is divisible by 9 is $45 \times 9 = 405$. Then 405, 414, 423, 432, 441, 450, 459, 468, 477, 486, 495, 504, 513, 522, 531, 540, 549, 558, 567, 576, 585, and 594 are all divisible by 9, 22 numbers in all. ***FOLLOW-UPS:*** *How many numbers between 401 and 599 are divisible (1) by 2 and 5? (2) by 2 or 5?* [19; 119]

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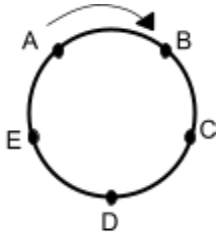
Exam for Grade 5

February 18, 2023

A SOLUTIONS	An ant travels around the circle in the direction shown. It touches each of the labeled points in order. The first three points that the ant touches are A, B, and C, in that order. What is the 28th point that the ant touches? 
	METHOD 1: *Strategy*: *Count by complete circuits of the circle.* The letter E is touched every 5 points beginning with the fifth point. Thus E is the 25th touch and **C is the 28th point the ant touches.** **METHOD 2:** *Strategy: Count by individual points.* The points in order are **ABCDE ABCDE ABCDE …** . The 28th point that the ant touches is C. ***FOLLOW-UPS:*** *(1) What is the 528th point the ant touches?* [C] *(2) Suppose the ant touches every second point, beginning with A. The first four points are A, C, E, B. What is the 28th point the ant touches? The 2006th?* [E, A]
B SOLUTIONS	If a three-digit number is divided by 5 or by 6, the remainder is 1 in each case. What is the least such three-digit number?
	METHOD 1: *Strategy: Make two lists and look for a common member.* Three-digit multiples of 5: 100, 105, 110, 115, **120**, … Three-digit multiples of 6: 102, 108, 114, **120**, … **The least number to leave a remainder of 1 when divided by 5 or by 6 is 121.** **METHOD 2:** *Strategy: Use the least common multiple.* The number sought is 1 more than a multiple of 5 and also 1 more than a multiple of 6. Since 5 and 6 have no factors in common, their least common multiple is $5 \times 6 = 30$. The smallest three-digit multiple of 30 is 120, so the least such number is 121. ***FOLLOW-UP:*** *What is the smallest number greater than 1 which leaves a remainder of 1 when divided by 2,3,4,5,6,7,8, or 9?* [2521]
C SOLUTIONS	Mr. Jackson was born on January 1, 1970. His daughter Lea was born January 1, 1992. In what year was Mr. Jackson exactly three times as Lea?

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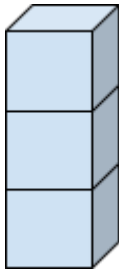
	Strategy: Start with the difference in their ages. 1992 –1970 = 22. Mr. Jackson is 22 years older than Lea. Then: **METHOD 1:** *Strategy: Express this difference in terms of Lea's age.* In the year in question, Mr. Jackson's age can be expressed as (Lea's age) + (Lea's age) + (Lea's age). The difference in their ages, 22 years, is then twice Lea's age, so Lea is 11 years old. Mr. Jackson is 11 + 22 = 33 years old. Eleven years after 1992 is 2003, as is 33 years after 1970. **The year was 2003.** **METHOD 2:** *Strategy: Make a chart listing their ages each year.* Mr. Jackson's age in the required year is 3 times Lea's age. The chart lists multiples of 3 for his ages and then subtracts 22 years to get her corresponding ages. | Mr. Jackson's age | 24 | 27 | 30 | **33** | | Lea's age | 2 | 5 | 8 | **11** | | Is Mr. J 3 times as old? | No | No | No | **Yes** | | Year | 1994 | 1997 | 2000 | **2003** | The only time that his age was three times hers was when he was 33 years and she was 11 years. The year was 2003. **METHOD 3:** *Strategy: Use algebra.* Let L be Lea's age when Mr. Jackson's age is 3 times as great. When Lea is L years old, Mr. Jackson's age can be expressed two ways: $L + 22$ and $3L$. Equate the two ways: $3L = L + 22$ Subtract L from each side of the equation: $2L = 22$ Divide each side of the equation by 2: $L = 11$ Lea is 11 years old in the year 1992 + 11 or 2003. ***FOLLOW-UP:*** *Dave is four times as old as Jeff. In 10 years, he will be twice as old. How old are they now?* [Dave is 20; Jeff is 5.]
D **SOLUTION**	How many 3-digit numbers are multiples of 21? **METHOD 1:** *Strategy: Find the least and greatest 3-digit multiples of 21.* (1) Because 999 ÷ 21 = 47 R12, there are 47 three-digit multiples of 21. (2) Because 100 ÷ 21 = 4 R16, there are 4 two digit multiples of 21. (3) Of the 47 multiples of 21 that are less than 1000, 4 of them have only 2 digits, so **there are 43 three-digit numbers that are multiples of 21.**
E **SOLUTION**	Three identical cubical boxes form a stack. It takes 350 sq cm of wrapping paper to completely wrap the whole stack with no overlap. Suppose each cube is wrapped separately and completely instead. What is the least amount of additional paper that is needed, in sq cm? 

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	Strategy: *Find the area of one face of a cube.* The surface of the stack consists of 14 squares (the top of the stack, the bottom of the stack, and the 12 vertical squares on the sides of the stack). The area of each square is $350 \div 14 = 25$ sq cm. If the cubes are wrapped separately, each cube has to be covered on 6 square faces, a total of $3 \times 6 = 18$ faces. That is 4 more faces than were covered originally and therefore $4 \times 25 =$ **100 sq cm of additional paper is needed.**  ***FOLLOW-UP:*** *Five cubes, whose edges measure 1,2,3,4, and 5 cm, respectively, are stacked so that each of them except the largest rests on top of the next largest one. How many sq cm of paint are needed to cover the exposed surface of the pile? (Include the bottom face of the largest cube.)* [270]

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Exam for Grade 6

February 18, 2023

A SOLUTION	The digits of a four-digit number are 1, 3, 6, and 9, but not necessarily in that order. The thousands digit is prime. The hundreds digit is 3 more than the tens digit. What is the number?
	Strategy: Fulfill one requirement at a time. 3 is the only prime number given .....3 _ _ _ 9 is 3 more than 6..... 3 9 6 _ 1 is the only digit left..... 3 9 6 1 **The number is 3,961.** (1 is not a prime number. See *What Every Young Mathlete Should Know.)*
B SOLUTION	The arithmetic mean (average) of five numbers is 8. Two of the numbers are 2 and 5. The other three numbers are equal. What is the value of one of the three equal numbers?
	Strategy: Find the sum of the five numbers. Because the average of the five numbers is 8, the sum of those five numbers is $5 \times 8 = 40$. The sum of the other three numbers is $40 - 2 - 5 = 33$. Then **the value of any one of the three equal numbers is** $33 \div 3 = \mathbf{11}$. ***Follow-Ups:*** *The average of five different counting numbers is 8. Consider the greatest of these numbers. (1)What is its greatest possible value?* [30] *(2) What is its least possible value?* [10]
C SOLUTION	The floor of a rectangular room is completely covered with square tiles. The room is 9 tiles long and 5 tiles wide. Find the number of tiles that touch the walls or door of the room.
	METHOD 2: *Strategy: Draw a diagram, and count the tiles that touch the edges.*  24 tiles touch the walls.
D SOLUTION	1 blue marble and 2 green marbles cost 16 cents. 1 red marble and 2 blue marbles also cost 16 cents.

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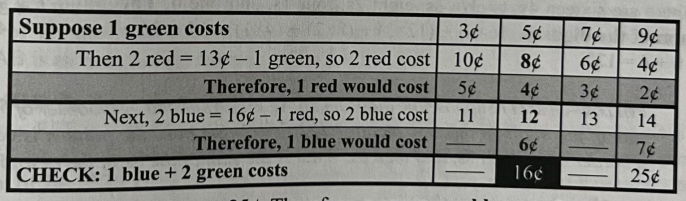
	1 green marble and 2 red marbles only cost 13 cents. How much does 1 green marble cost?
	METHOD 1: *Strategy: Set up a table. Use number properties to limit choices.* The cost of 1 green plus 2 red marbles is an odd number of cents but the cost of 2 red marbles is even. So the cost of 1 green marble is odd. The top row of the table lists each possible cost of 1 green. The other rows use this number to figure the cost of 1 red and 1 blue marble.  1 blue and 2 green cost 16¢, not 25¢. Therefore, **a green marble costs 5 cents.** **METHOD 2:** *Strategy: Combine the given information.* Suppose all 3 purchases are made. Then 3 green marbles, 3 blue marbles, and 3 red marbles cost a total of 45 cents. So 1 green marble, 1 blue marble, and 1 red marble cost 15 cents. The first and second sentences show that 1 blue and 2 green marbles cost as much as 1 red and 2 blue marbles. If we imagine a balance scale with 1 blue and 2 green marbles on one side and 1 red and 2 blue marbles on the other, we could remove 1 blue marble from each side and see that 2 green marbles balance 1 red and 1 blue marble. Thus, we could replace 1 red and 1 blue with 2 green marbles. So we know that 3 green marbles cost 15 cents, so each green marble costs **5 cents**. ***FOLLOW-UP:*** *At a movie theater, 2 popcorns and a soda cost $13, while 5 popcorns and 4 sodas cost $37. Julia orders a popcorn and a soda. How much does Julia spend?* [$8]
E **SOLUTION**	What is the least prime number that is a divisor N? N = (13 × 17) + (19 × 23)
	METHOD 1: *Strategy: Determine whether the result is even or odd.* Find a way to avoid all that computation. The product of two odd numbers is odd, and the sum of two odd numbers is even. Therefore, $(13 \times 17) + (19 \times 23)$ is even. The least prime number, 2, is a factor of every even number. **The smallest prime number that divides the given expression is 2.** **METHOD 2:** *Strategy: Do the arithmetic and then factor.* $(13 \times 17) + (19 \times 23) = 221 + 437 = 658$. $658 = 2 \times 7 \times 47$. Observe that once the factor of 2 is found, it is not necessary to factor further: the least prime number that divides the sum is 2. ***FOLLOW–UPS:*** *(1) What is the greatest prime number that divides $(17 \times 13) + (17 \times 23)$?* [17] *(2) What is the greatest prime number that divides $(17 \times 25) + (17 \times 20) + (17 \times 12)$?* [19; the expression = 17 × 57] *(3) What prime numbers divide $(14 \times 31) - (7 \times 17)$?* [7, 5, 3; think $14 \times 31 = 7 \times 62$]

Suppose 1 green costs	3¢	5¢	7¢	9¢
Then 2 red = 13¢ – 1 green, so 2 red cost	10¢	**8¢**	6¢	4¢
Therefore, 1 red would cost	5¢	**4¢**	3¢	2¢
Next, 2 blue = 16¢ – 1 red, so 2 blue cost	11	**12**	13	14
Therefore, 1 blue would cost	——	**6¢**	——	7¢
CHECK: 1 blue + 2 green costs	——	16¢	——	25¢

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Exam for Grade 7

February 18, 2023

A SOLUTION	Zach buys two hot dogs and three drinks for $14. Drinks cost $2 each. How much does one hot dog cost?
	METHOD 1: *Strategy: Subtract the cost of the drinks.* Three drinks cost $6. Therefore 2 hot dogs cost $8. **One hot dog costs $4.** **METHOD 2:** *Strategy: Use algebra* Let h represent the price of a hot dog. Then: 2 hot dogs plus 3 $2-drinks cost $14. Multiply $2 by 3 — $2h + 3(2) = 14$ Subtract $6 from each side of the equation : — $2h + 6: = 14$ Divide each side of the equation by 2 — $2h = 8$, $h = 4$ One hot dog costs $4.
B SOLUTION	Staci looks at the first and fourth pages of a chapter in her book. The sum of their page numbers is 47. On what page did the chapter begin?
	METHOD 1: *Strategy: Find the average of the page numbers.* The sum of the first and fourth page numbers is 47. This is also the sum of the second and third page numbers. The average of the 4 page numbers is $23\frac{1}{2}$. $23\frac{1}{2}$ is immediately between the second and third page numbers. The page numbers are 22, 23, 24, and 25. **The chapter begins on page 22.** **METHOD 2:** *Strategy: Use algebra.* Let P represent the first page number. The other numbers are $P + 1$, $P + 2$, and $P + 3$. Then $P + (P + 3) = 47$. Solving, $P = 22$. The chapter begins on page 22.
C SOLUTION	The sum of three whole numbers, *A, B,* and *C*, is 32. *C* is 10 more than *A*. *B* differs from one of the other numbers by 3 and the other by 7. Find *B*.
	Strategy: Set up and solve possible equations. C is 10 more than A. Also, B. is 3 away from one number and 7 away from the other number. Then $A < B < C$. Moreover, either $B = A + 3$ or $B = A + 7$ as shown below. 3, 7 — A B C 7, 3 — A B C Substitute $C = A + 10$ and each possibility for B into $A + B + C = 32$. The result will be two equations: $A + (A + 3) + (A + 10) = 32$ and $A + (A + 7) + (A + 10) = 32$. The first equation results in $A = \frac{19}{3}$ and the second equation results in $A = 5$. A is a whole number so $A = 5$ and $\boldsymbol{B = 12}$.

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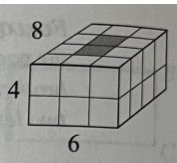
D SOLUTION	Each of Mia's marbles show several colors. $\frac{2}{5}$ of the marbles show some red. $\frac{3}{4}$ of the marbles show some yellow. $\frac{6}{7}$ of the marbles show some blue. Mia has fewer than 250 marbles. How many of Mia's marbles show some blue?
	Strategy: Find the total number of marbles. Since $\frac{2}{5}$ of the marbles show some red, the number of marbles is a multiple of 5. Likewise, the total number of marbles is also a multiple of 4 and of 7. The least common multiple of 4, 5, and 7 is 140. The next common multiple is 280, which is too large. Mia has 140 marbles. Because $\frac{6}{7}$ of 140 is 120, **120 marbles show some blue.**
E SOLUTION	A rectangular solid that is 4cm by 6cm by 8cm is painted on all six faces. Then the solid is cut into cubes that measure 2 cm on each side. How many of these cubes have only one face painted?
	Strategy: Draw a diagram. Draw the rectangular solid showing how it was cut into 2-cm cubes. Eliminate the 8 corner cubes (3 faces painted) and the 12 edge cubes (2 faces painted.) **4 of these cubes have only one face painted.**  ***Follow-Ups:*** *Suppose the rectangular solid in 4E is cut into 1-cm cubes. (1) How many cubes have three faces painted?* [8] *(2) No faces painted?* [48] *(3) Into how many 1-cm by 2-cm by 3-cm rectangular solids can the figure in 4E be cut?* [32]

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Exam for Grade 8

February 18, 2023

A **SOLUTION**	How many digits are in the product of the following: 2 X 3 X 5 X 2 X 3 X 5 X 2 X 3 X 5
	Strategy: Group the factors to simplify the multiplication. **METHOD 1:** $(2\times3\times5)\times(2\times3\times5)\times(2\times3\times5) = 30\times30\times30 = 27{,}000$. **There are 5 digits in the product.** **METHOD 2:** $3\times3\times3\times(2\times5)\times(2\times5)\times(2\times5) = 27\times10\times10\times10 = 27{,}000$. There are 5 digits in the product. ***FOLLOW-UP:*** *The number 10,000 is written as the product of two numbers, neither of which has 10 as a factor. What is the sum of these two numbers?* [641]
B **SOLUTION**	A toll bridge charges $4 for a car and $6 for a truck. One day 200 of these vehicles crossed the bridge and paid a total of $860 in tolls. How many of these vehicles were trucks?
	METHOD 1: *Strategy: Start with a specific number of each vehicle.* Suppose all 200 vehicles were cars. The toll total would be $4 \times 200 = \$800$, which is $60 too low. Each car that is replaced by a truck increases the toll total by $2. To increase the total by $60, replace $60 \div 2 = 30$ cars by trucks. Then **30 of the vehicles were trucks.** Check the answer: $(30 \times \$6) + (170 \times \$4) = \$860$. **METHOD 2:** *Strategy: Use algebra.* Let T = the number of trucks. They paid a total of $6T$ dollars. Then $200 - T$ = the number of cars. They paid a total of $4(200 - T)$ dollars. Together, all the cars and trucks paid a total of $860. $4(200 - T) + 6T = 860$ Multiply $(200 - T)$ by 4: $800 - 4T + 6T = 860$ Add $6T$ to $-4T$: $800 + 2T = 860$ Subtract 800 from each side of the equation: $2T = 60$ Divide each side of the equation by 2: $T = 30$ 30 of the vehicles were trucks. Check as in method 1.
C **SOLUTION**	Group A has 10 numbers with an average of 10. Group B has 20 numbers with an average of 20. Group C has 30 numbers with an average of 30. Group D has 40 numbers with an average of 40. The four groups are combined into a single group. What is the average of the combined group?

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Strategy: *Use the definition of average.*
The average of a group of numbers is their total divided by the number of numbers, so the product of the number of numbers in the group and their average is equal to their total.

Group	Quantity × Average	= Total in group
A	10 × 10	= 100
B	20 × 20	= 400
C	30 × 30	= 900
D	40 × 40	= 1600
	Total of all groups	= 3000

The average of the combined group is $3000 \div 100 = \mathbf{30}$.

FOLLOW-UP: *Jimmy bikes to a park, a distance of 30 miles, at 10 miles per hour. He bikes home along the same path at 6 miles per hour. What is Jimmy's average speed for the entire trip?* [7.5 miles per hour]

D
SOLUTION

How many 3-digit numbers have exactly 2 digits that are the same?

METHOD 1: *Strategy*: *Break the problem into cases.*
Numbers with exactly two digits the same look like ABB or BAB or BBA, where A and B represent two different digits.

(a) First consider numbers of the form ABB. A may not equal 0 in this arrangement. If $A = 1$, there are nine such numbers (100, 122, 133, …, 199). There are also nine such numbers for each of $A = 2, 3, 4, \ldots, 9$. Thus, there is a total of $9 \times 9 = 81$ numbers of the form ABB.

(b) Consider numbers of the form BAB. If $A = 0$, there are nine such numbers (101, 202, 303, …, 909). Since B may not equal 0, there are only eight such numbers for each of $A = 1, 2, 3, \ldots, 9$. Thus, there is a total of $9 + 8 \times 9 = 81$ numbers of the form BAB.

(c) A similar argument shows that there is also a total of 81 numbers of the form BBA.

Therefore, a total of $3 \times 81 =$ **243 three-digit numbers have exactly two digits the same.**

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METHOD 2: *Strategy: Count the numbers that don't have exactly two digits the same.*
The digits of a 3-digit number are in 3 classes. Either all are the same, just two of them are the same, or all are different.

(a) **All the same:** These numbers have the form AAA. There are 9 such numbers (111, 222, 333, …, 999).

(b) **All different:** These numbers have the form ABC. There are 9 choices for the digit A, since A cannot be 0. For each value of A, there are 9 choices left for B (since B could be 0). For each pair of values of A and B, 8 choices remain for C. Thus, there are $9 \times 9 \times 8 = 648$ numbers of the form ABC.

(c) Since there are 999 numbers less than 1000, of which 99 have two digits, there are 900 three-digit numbers. Then $900 - 648 - 9 = 243$ three-digit numbers have exactly two identical digits.

METHOD 3: *Strategy: Make a list.*

Repeated digit	List of numbers	Quantity	
2 zeros	**100**, 200, 300, …, 900		**9**
2 ones	**110**, 112, 113, …, 119 (*not 111*)	9	
	101, 121, 131, …, 191 (*not 111*)	9	**26**
	211, 311, 411, …, 911 (*not 111*)	8	
2 twos	**220**, 221, 223, …, 229 (*not 222*)	9	
	202, 212, 232, …, 292 (*not 222*)	9	**26**
	122, 322, 422, …, 922 (*not 222*)	8	

As shown in the table, 2 zeroes appear in nine numbers. Also, 2 ones and 2 twos each appear in another 26 numbers. Similarly, 2 threes, 2 fours, …, and 2 nines each appear in 26 numbers. This is a total of $9 + (26 \times 9) = 243$ numbers. In all, there are 243 three-digit numbers with two identical digits.

E
SOLUTION

What number exceeds 23 by 4 more than twice the amount by which 15 exceeds -3?

Strategy: Work backwards from the end of the sentence.

What number exceeds 23 by 4 more than twice **the amount by which 15 exceeds –3**?

What number exceeds 23 by 4 more than twice **18** ?

What number exceeds 23 by 4 more than **36** ?

What number exceeds 23 by **40** ?

The number is 63.