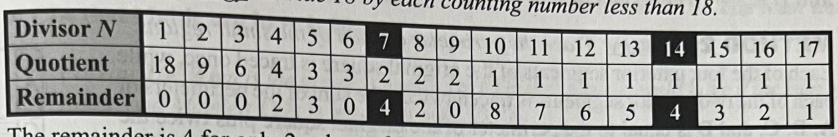
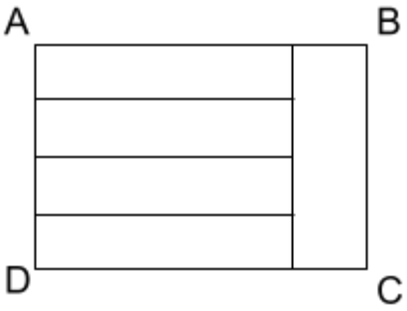
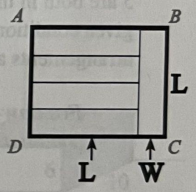
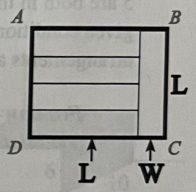
Exam for Grade 4

February 12, 2023

<table>
<tr><td>A
SOLUTION</td><td>Choose any number between 32 and 56.
Add 20. Subtract 17. Add 13. Subtract your original number.
What is the resulting number?</td></tr>
<tr><td></td><td>METHOD 1: Strategy: Rearrange the numbers to simplify the problem.
Since the only operations are addition and subtraction, the number chosen is subtracted out. It is only necessary to start with 20, subtract 17 and add 13.
$\square + 20 - 17 + 13 - \square = 20 - 17 + 13 = 16$. The result is 16.
METHOD 2: Strategy: Choose a number between 32 and 56.
Choose any number (such as 40) between 32 and 56 and perform the indicated operations:
$40 + 20 - 17 + 13 - 40 = 16$.
It is wise to repeat, choosing at least two other numbers, in order to make sure 16 is the only result regardless of the number you choose.</td></tr>
<tr><td>B
SOLUTION</td><td>When 18 is divided by the whole number N, the remainder is 4.
This is true for how many different values of N?</td></tr>
<tr><td></td><td>METHOD 1: Strategy: Consider one restriction on N at a time.
N satisfies two conditions: (1) N is a factor of 14; and (2) $N > 4$, because a remainder is less than the divisor. The factors of 14 are 1, 2, 7, and 14. Of these only 7 and 14 are greater than 4. Therefore, there are 2 values of N.
METHOD 2: Strategy: Divide 18 by each counting number less than 18.
 
<table>
<tr><td>Divisor N</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td><td>13</td><td>14</td><td>15</td><td>16</td><td>17</td></tr>
<tr><td>Quotient</td><td>18</td><td>9</td><td>6</td><td>4</td><td>3</td><td>3</td><td>2</td><td>2</td><td>2</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>Remainder</td><td>0</td><td>0</td><td>0</td><td>2</td><td>3</td><td>0</td><td>4</td><td>2</td><td>0</td><td>8</td><td>7</td><td>6</td><td>5</td><td>4</td><td>3</td><td>2</td><td>1</td></tr>
</table>
The remainder is 4 for only 2 values of N, 7 and 14.
Follow-Up: Suppose 50 is divided by each counting number less than 50. What is the least remainder that occurs only once? [3; the divisor is 47.]</td></tr>
<tr><td>C
SOLUTION</td><td>Rectangle ABCD is divided into five congruent smaller rectangles, as shown. The lengths BC and CD differ by 6 millimeters. What is the area of ABCD?</td></tr>
</table>

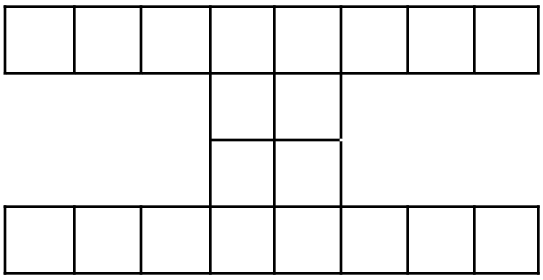
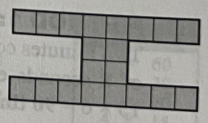
	
	Strategy: Find the dimensions of the small rectangles. Represent BC by L and DC by $L + W$. Then $DC - BC = W$. $DC - BC = 6$, so $W = 6$ mm and $AD = 4W = 24$ mm. **The area of *ABCD* is** $24 \times (24 + 6) =$ **720 sq mm.**  ***FOLLOW-UP:*** *Determine the difference in the lengths BC and CD given that the area of rectangle ABCD is 2000 square mm. [10 mm]*
D SOLUTION	Janine's number has three digits. One digit is a prime number. Another digit is a square number. The other digit is neither prime nor square. Her number is NOT divisible by 3. What is the greatest possible value of Janine's number?
	Strategy: Find the dimensions of the small rectangles. Represent BC by L and DC by $L + W$. Then $DC - BC = W$. $DC - BC = 6$, so $W = 6$ mm and $AD = 4W = 24$ mm. **The area of *ABCD* is** $24 \times (24 + 6) =$ **720 sq mm.**  ***FOLLOW-UP:*** *Determine the difference in the lengths BC and CD given that the area of rectangle ABCD is 2000 square mm. [10 mm]*
E SOLUTION	A bowl contains 100 pieces of colored candy: 48 green, 30 red, 12 yellow, and 10 blue. They are all wrapped in foil, so you do not know the color of any piece of candy. What is the least number of pieces you must take to be certain that you have at least 15 pieces of the same color?

	Strategy: Consider the worst case. Determine the largest number of pieces you could take and still not have 15 of the same color. You could take all 12 yellow, all 10 blue, 14 of the green, and 14 of the red pieces and still not have 15 of the same color; so far you have a total of 50 pieces. The next piece you take, whether green or red, gives you 15 matching pieces. **The least number of pieces you must take to be sure that you have 15 pieces of the same color is 51.** ***FOLLOW-UP:*** *Ana has 8 pennies, 3 quarters, 6 nickels, and 5 dimes in her piggy bank. She needs a dollar to buy a card. What is the greatest number of coins Ana can take out of the bank and still not have enough for the card?* [19] *What is the fewest number of coins that will get her the card?* [6]

Exam for Grade 5

February 12, 2023

A SOLUTION	What is the value of (47 × 8) + (8 × 27) + (26 × 8)?
	A **METHOD 1:** *Strategy: Use the distributive property of multiplication over addition.* $(47 \times 8) + (8 \times 27) + (26 \times 8) = (47 + 27 + 26) \times 8 = 100 \times 8 = \mathbf{800.}$ **METHOD 2:** *Strategy: Perform the operations as indicated.* $(47 \times 8) + (8 \times 27) + (26 \times 8) = 376 + 216 + 208 = 800.$ **3** *Strategy: Maximize the minuend; minimize the subtrahend.* Use the three greatest digits in the minuend and arrange them from greatest to least. Arrange the other three digits from least to greatest in the subtrahend. Then **531 is the greatest possible difference.** 654 − 123 = 531 ***FOLLOW-UP:*** *The ten digits are each used once to form two 5-digit even numbers whose difference is a maximum. What are the numbers?* [98,756 – 10,234 = 88,522]
B SOLUTION	In the repeating pattern below, what will be the 78th letter written? ABBCCD ABBCCD….. (and so on)
	Strategy: Determine the number of letters in the repeating part. In the sequence ABBCCDABBCCD…, there are 6 letters before the pattern repeats itself. The first 78 terms of the sequence contain 13 complete repetitions of the pattern and no additional letters. **The 78th letter is** the same as the 6th letter, which is **D**. *FOLLOW-UPS: (1) In the sequence of letters ABBCCCDDDDEEEEEFFFFFF…, what is the 100th letter?* [N] *(2) A letter is chosen in FOLLOW-UP (1) from the first 50 letters. What is the probability that it is a vowel?* [$\frac{15}{50}$ or $\frac{3}{10}$]
C SOLUTION	Four volunteers can pack 12 boxes every 30 minutes. How many additional volunteers are needed to pack 72 boxes every hour? [Assume all volunteers work at the same rate.]

	METHOD 1: *Strategy: Find how much 4 volunteers can do in 1 hour.* If 4 volunteers can pack 12 boxes every 30 minutes, then 4 volunteers can pack 24 boxes every hour (60 minutes). In order to pack 72 boxes in an hour, since $72 \div 24 = 3$, three times as many volunteers are needed. So 12 volunteers are needed to do the job; therefore, **8 additional volunteers are needed.** **METHOD 2:** *Strategy: Find how many boxes 1 volunteer can pack in an hour.* If 4 volunteers can pack 12 boxes every 30 minutes, then 1 volunteer can pack $\frac{1}{4}$ as many in 30 minutes, or 3 boxes. One volunteer can then pack 6 boxes in an hour. To pack 72 boxes requires 12 volunteers, and so 8 additional volunteers are needed. ***Follow-Up:*** *If 8 volunteers can pack 12 boxes in 30 minutes, how long would it take 6 volunteers to pack 24 boxes?* [80 minutes]
D **Solution**	Each small region in the figure shown is a square. The area of the entire figure is 320 sq cm. What is the number of cm in the perimeter of the entire figure? 
	Strategy: Determine the length of the side of a small square. The figure consists of 20 small congruent squares. The area of each small square is $\frac{1}{20}$ of the total area of 320 sq cm, which is 16 sq cm. Then the length of a side of each small square is 4 cm. **The perimeter of the entire figure** consists of 36 sides, so it **is** $36 \times 4 =$ **144 cm.** 
E **Solutions**	The pages of a book are numbered consecutively, beginning with 1. The digit 7 is printed 25 times in numbering the pages. What is the largest number of pages the book can have?

Strategy: Count the number of 7s in an organized way.

METHOD 1: From pages 1 through 100, the digit seven appears 10 times as a units digit and 10 times as a tens digit (the 70s) for a total of 20 times. The remaining 5 sevens are used to number pages 107, 117, 127, 137, and 147. A 26th seven would be required to number page 157. Therefore, **the largest number of pages the book can have is 156.**

METHOD 2: This table separates the numbers in which 7 appears in the tens place from the rest of the numbers. Adjust the cumulative total once it passes 25 sevens.

pages	Number of Sevens			
	units place	tens place	subtotals	cum.
1 to 69	7	0	7	7
70 to 79	1	10	11	18
80 to 169	9	0	9	27

Numbering up to page 169 requires 27 sevens. This is two more than are available. The two pages less than 169 that should not be numbered are pages 167 and 157. The largest number of pages the book can have is 156.

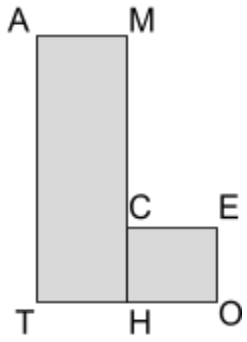
Exam for Grade 6

February 12, 2023

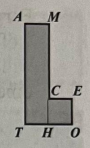
A SOLUTIONS	What is the greatest number of Mondays that can occur in 45 consecutive days?
	METHOD 1: *Strategy: Use the definition of a week.* In 45 consecutive days there are 6 weeks and 3 days. Each of the 6 weeks contains one Monday. In order to have the greatest number of Mondays, one of the 3 days left must also be a Monday. **The greatest number of Mondays that can occur in 45 consecutive days is 7.** **METHOD 2:** *Strategy: Start at 1 and count by sevens.* Suppose day 1 is a Monday. Mondays will occur on days 1, 8, 15, 22, 29, 36, 43. The next Monday would be after day 45. The greatest number of Mondays in 45 consecutive days is 7. ***FOLLOW-UP:*** *Today is Tuesday. What day is it 100 days from now? 1000?* [Thursday; Monday]
B SOLUTIONS	Mrs. Saada is between 50 and 80 years old. If you divide her age by 9, the remainder is 1. If you divide her age by 4, the remainder is 1. How old is Mrs. Saada?
	METHOD 1: *Strategy: Consider Mrs. Saada's age last year.* Last year her age was a multiple of both 9 and 4 and thus a multiple of 36. The only multiple of 36 between 50 and 80 is 72. Then this year **Mrs. Saada is 73 years old.** **METHOD 2:** *Strategy: List the numbers that satisfy one of the conditions.* It is faster to divide by 9 than by 4 because 9 produces fewer results. The only numbers between 50 and 80 that are 1 more than a multiple of 9 are 55, 64, and 73. Of these, only 73 is one more than a multiple of 4. Mrs. Saada is 73 years old. ***FOLLOW-UP:*** *Kai and his mother are younger than 10 and 50 years old, respectively. When the sum of their ages is divided by 5, the remainder is 2. When each of their ages is divided by 7, there is no remainder. How old is Kai's mother?* [35]

C SOLUTION

The area of the rectangle MATH is 30 sq cm and each side length is a counting number of cm. H is the midpoint of $\overline{TO}$. The area of square ECHO is between 5 sq cm and 24 sq cm. Find the perimeter of the entire figure, in cm.



METHOD 1: *Strategy: Start with the area of MATH.*
Rectangle *MATH* is either 1 by 30, 2 by 15, 3 by 10, or 5 by 6 cm. *TH* is one of these eight lengths.



The area of square *ECHO* is between 5 and 24, and *TH* = *HO*. Then *HO* can only be 3 because each of the other seven choices produces an area that is too large or too small. Then *TH* = 3, *HO* = 3, *OE* = 3, *EC* = 3, *CH* = 3, *AM* = 3, *AT* = 10. Then *MH* = 10, so *MC* = 7.

The perimeter of the shaded figure, *MA* + *AT* + *TH* + *HO* + *OE* + *EC* + *CM*, **is 32 cm.**

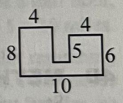
METHOD 2: *Strategy: Start with the area of ECHO.*
The area of *ECHO* could be 9 or 16 sq cm, so *HO* would be 3 or 4 cm. Since *H* is the midpoint of $\overline{TO}$, *TH* is either 3 or 4 cm.

The area of *MATH* is 30 sq cm. Of 3 and 4, only 3 is a factor of 30, so each side of *ECHO* is 3 cm long.

Thus, *TH* = 3 cm and *AT* = 10 cm. Further, *MA* = 3 and *MC* = 10 – 3 = 7 cm.

Then the perimeter of the shaded figure is 32 cm, which is the sum of its sides.

FOLLOW-UPS: *All lengths in the figure are in cm and all angles are right angles. (1) What is the perimeter of the figure?* [46 cm] *(2) What is its area?* [58 sq cm]



D SOLUTIONS

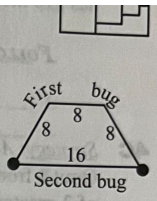
If 4 people can paint 2 fences in 5 hours, how many hours in all will it take for 8 people to paint 8 fences?

METHOD 1: *Strategy: First adjust the number of fences.*
Two fences can be painted by 4 people in 5 hours, so 8 fences takes those same 4 people 4 times as long, or 20 hours. If 4 people need 20 hours to paint the 8 fences, then 8 people will take only half as long to do the job. **It takes 10 hours for 8 people to paint 8 fences.**

METHOD 2: *Strategy: First adjust the number of people.*
If 4 people need 5 hours to paint the 2 fences, then 8 people need only half as long ($2\frac{1}{2}$ hours) to paint the 2 fences. To paint 8 fences, the 8 people need 4 times as long, 10 hours.

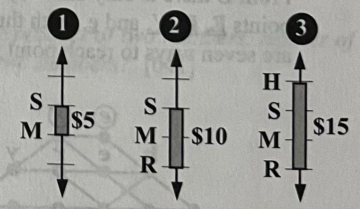
E

Two bugs walk from point A to point D along the sides of figure

SOLUTION	ABCD. They start and finish together. The first bug walks from A to B to C to D at an average speed of 3 cm per second. The second bug walks directly from A to D. What is the average speed of the second bug?
	METHOD 1: *Strategy: Compare the distances the two bugs walked.* The second bug covers 16 cm in the same time the first bug covers 24 cm. Because the second bug covers only $\frac{2}{3}$ the distance of the first bug, it travels at $\frac{2}{3}$ the speed. The speed of the first is 3 cm per sec. **The speed of the second bug is 2 cm per sec.**  **METHOD 2:** *Strategy: Find the time that each bug traveled.* The first bug needed 8 seconds to travel the 24 cm. Its rate was 3 cm per second. The second bug traveled 16 cm in the same 8 seconds, so its rate was 2 cm per second.

Exam for Grade 7

February 12, 2023

A SOLUTION	Chloe divides the number N by 14 correctly and gets 5. Mia misreads the division sign as an addition sign and adds 14 to N. What sum does Mia get?
	Olympiad 4 *Strategy: Work backwards.* Before Chloe divides the number N, it is 14 times as large as her final answer, 5. So, N is equal to $14 \times 5 = 70$. When Mia accidentally adds the 14 to the 70, **Mia gets an answer of 84.**
B SOLUTION	Michael has \$5 less than Samantha. Samantha has \$10 more than Rob. Rob has \$15 less than Hailey. How many more dollars does Hailey have than Micheal?
	METHOD 1: *Strategy: Draw a diagram to compare their amounts.* Represent each person by an initial. Rank them from most to least, top to bottom. 1. Michael has \$5 less than Samantha. 2. Samantha has \$10 more than Rob. 3. Rob has \$15 less than Hailey.  The picture shows that **Hailey has \$10 more than Michael.**
	METHOD 2: *Strategy: Assign a convenient amount to Michael.* Suppose Michael has \$50. Then Samantha has \$55. Rob has \$45 and Hailey has \$60. Hailey has \$10 more than Michael. ***Follow-Up:*** *In Method 2, choose other amounts for Michael. Why does this method work?*
C SOLUTION	A sports arena has a total capacity of 20,000 fans and ushers. One usher is required for every 30 fans. What is the greatest number of fans that can be in attendance?
	Strategy: Minimize the number of ushers. Group each 30 fans with 1 usher to form groups of 31. Then the 20,000 people are divided into 645 groups of 31 each, with 5 people left over. Those 5 people must contain at least 1 usher and at most 4 fans. There must be at least $645 + 1 = 646$ ushers. **There are at most** $20{,}000 - 646 = 19{,}354$ **fans that can be in attendance.**

D **SOLUTION**	Bert has 40% more jelly beans than Vicki. What fractional part of Bert's jelly beans must be given to Vicki so that they each have the same number of jelly beans? Express your answer in lowest terms.
	METHOD 1: *Strategy: Convert the percent to a fraction.* 40% of a number is $\frac{2}{5}$ of that number. Bert has $\frac{2}{5}$ *more* jelly beans than Vicki. That is, for every 5 jelly beans that Vicki has, Bert has $5 + 2 = 7$ of them. So of every 7 jelly beans that Bert has, he must give 1 to Vicki in order for them to have the same number. **Bert must give $\frac{1}{7}$ of his jelly beans to Vicki.** **METHOD 2:** *Strategy: Pick a convenient starting number.* Suppose Vicki has 100 jelly beans. Then Bert has 40% more, or 140 jelly beans. If Bert gives 20 jelly beans to Vicki, they will each have 120. Bert has to give $\frac{20}{140} = \frac{1}{7}$ of his jelly beans to Vicki. To verify, assign at least two other numbers to Vicki to start with. ***Follow-Ups:*** *(1) A store decreases the price of a coat by 40%, and then increases the price by 40%. What percent of the original price is the final price of the coat?* [84 or 84%] *(2) Why is the final price less than the original price?* [The original price is higher than the discounted price, so 40% of the higher price will be greater than 40% of the lower price. That is, more money was subtracted during the first transaction than was added during the second transaction.]
E **SOLUTION**	Larry starts at the bottom of a long staircase. He climbs exactly ⅔ of the stairs. Then, he goes back down exactly ½ of the way to the bottom. From that spot, he climbs exactly ⅔ of the way to the top. Finally, from there, he climbs 6 stairs to reach the top. How many stairs are in the staircase?

METHOD 1: *Strategy: Draw a diagram to show each action Larry takes.*

Larry climbs exactly $\frac{2}{3}$ of the steps. [Action A in method 2]

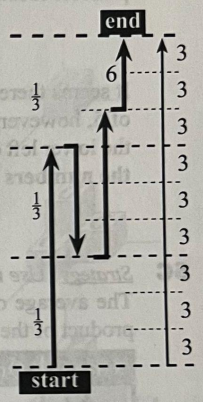
Then he goes back down exactly $\frac{1}{2}$ of the steps he just climbed:
$\frac{1}{2}$ of $\frac{2}{3} = \frac{1}{3}$. [Action B]

From that spot, he climbs exactly $\frac{2}{3}$ of the steps above him:
$\frac{2}{3}$ of $\frac{2}{3} = \frac{4}{9}$. [Action C]

From there, he climbs 6 stairs to reach the top.
Those 6 steps equal $\frac{2}{9}$ of the staircase.

Therefore, $\frac{1}{9}$ of the staircase = 3 steps and $\frac{9}{9}$ (the whole staircase) = 27 steps.

There are 27 stairs in the staircase.



METHOD 2: *Strategy: Guess and check; make a table.*

The total number of steps has to be divisible by 3 and 9, and greater than 9. Start with a total of 18, 27, 36, …, steps, until the number in the last column equals the starting number.

Guess	After A	After B	Steps remaining	After C	6 more steps
18	step 12	step 6	12	14	20 (too much)
27	step 18	step 9	18	21	**27**

The staircase has 27 steps.

Exam for Grade 8

February 12, 2023

A **SOLUTION**	A total of 20 marbles are placed into 5 cups. Each cup has a different number of marbles. No cup has exactly 4 marbles, and no cup is empty. What is the greatest number of marbles that any one cup can have?
	Strategy: Minimize the other numbers. To put as many marbles as possible in one cup, place as few as possible in the other cups. The four cups with the least number contain $1 + 2 + 3 + 5 = 11$ marbles. With a total of 20 marbles, **the greatest number any one cup can have is 9.**
B **SOLUTION**	Find the least positive integer A so that the product of 45 and A is a perfect square number.
	Strategy: Factor the number into prime numbers. $45 = 3 \times 3 \times 5$. A perfect square can be expressed as the product of two equal factors. $45 \times A = 3 \times 3 \times 5 \times A$. **The least positive integral value of A is 5.** ***Follow-Ups:*** *(1) What is the next least integer A for which $45 \times A$ is a perfect square?* [20] *(2) What is the least positive integer A for which $45 \times A$ is a perfect cube?* [75] *(3) What two years in the eleventh century (1001 to 1100) are perfect squares?* [1024,1089]
C **SOLUTION**	Joshua has more than 250 toy soldiers. When he tries to arrange them in rows of 3, there are 2 left over. When he tries to arrange them in rows of 5, there are 2 left over. When he tries to arrange them in rows of 7, there are 2 left over. What is the least number of toy soldiers Joshua may have?

METHOD 1: *Strategy: List the numbers that satisfy each condition.*
Numbers that are greater than 250 and 2 more than a multiple of:

7: 254, 261, 268, 275, 282, …, 310, **317**, 324, …
5: 252, 257, 262, 267, 272, 277, 282, …, 307, 312, **317**, 322, …
3: 251, 254, 257, 260, 263, 266, …, 314, **317**, 320, …

The least number of toy soldiers that Joshua may have is the least number to appear in all 3 lists, **317**.

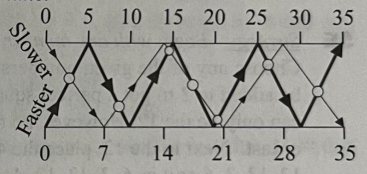
METHOD 2: *Strategy: Use common multiples.*
In each case 2 toy soldiers were left over. Suppose Joshua had 2 fewer toy soldiers. Then their number would be a multiple of 3, of 5, and of 7. No two of these numbers have a common factor, so their least common multiple is their product, 105. The number of soldiers is thus 2 more than a multiple of 105. The first multiple of 105 that is more than 250 is 315, so the least number of toy soldiers Joshua may have is $315 + 2 = 317$.

FOLLOW-UPS: *(1) What is the least four-digit number divisible by 5, 7, 15, and 21?* [1050] *(2) What is the least number that leaves a remainder of 3 when divided by 5, a remainder of 4 when divided by 6, and a remainder of 5 when divided by 7? (Hint: Find a nearby number that is a multiple of 5, 6, and 7.)* [208]

D
SOLUTION

Starting at the same time on opposite shores of a lake, two boats cross back and forth for 35 minutes without stopping. One boat needs 5 minutes to cross the lake. The other boat needs 7 minutes. What is the number of times during the 35 minutes that the faster boat passes the slower boat going in the same or opposite direction?

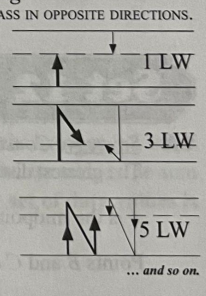
METHOD 1: *Strategy: Draw a diagram showing time.*
Mark the opposite shores in minutes, 0 to 35. The slower boat touches alternating shores at 7, 14, 21, 28, and 35 minutes. The faster boat touches alternating shores at 5, 10, 15, 20, 25, 30, and 35 minutes. Where the paths cross, the boats are passing. From the diagram, **the faster boat passes the slower boat 7 times.**



METHOD 2: *Strategy: Examine the distances covered at each passing.*
The first time the boats pass each other, the sum of the distance covered by both boats is 1 lake-width (LW). Each boat covers a fraction of a LW.

The second time the boats pass, going in *opposite* directions, the sum of the distances covered is 3 LWs. Each covers one-and-a-fraction LWs.

Similarly, the boats pass each other going in *opposite* directions when the sum of the distances is 5, 7, 9, … LWs. In 35 minutes the fast boat crosses the lake 7 times and the slow boat 5 times; they travel a total of 12 LWs. The boats pass going in opposite directions at 1, 3, 5, 7, 9, and 11 LWs, a total of 6 times.



E **SOLUTION**	The average of 6 consecutive odd numbers in 50. What is the least of these numbers?

METHOD 1: *Strategy: Work from the middle outward.*
Consider the numbers listed in order from least to greatest.

Since 50 is the average, and all numbers are equally spaced, the two consecutive odd numbers in the middle are 49 and 51.

Once these are in place, write the odd numbers that precede 49 and that follow 51 to complete the list.

__, __, __, __, __, __

__, __, 49, 51, __, __

45, 47, 49, 51, 53, 55

The least of these numbers is 45.

METHOD 2: *Strategy: Add the same amount to each member of the simplest possible set.*
The sum of any 6 numbers is equal to 6 times their average, so the sum of the required numbers is $6 \times 50 = 300$. Suppose the six consecutive odd numbers were 1, 3, 5, 7, 9, and 11. Their sum would be 36, which is 264 short of 300. Add $264 \div 6 = 44$ to each of 1, 3, 5, …, 11: The least number in the set {45, 47, 49, 51, 53, 55} is 45.